

Centre de Physique Théorique* - CNRS - Luminy, Case 907
F-13288 Marseille Cedex 9 - France

**LANDAU HAMILTONIANS
WITH RANDOM POTENTIALS :
LOCALIZATION
AND THE DENSITY OF STATES**

J.M. Combes^{1,2} and P.D. Hislop^{3,4}

Abstract

We prove the existence of localized states at the edges of the bands for the two-dimensional Landau Hamiltonian with a random potential, of arbitrary disorder, provided that the magnetic field is sufficiently large. The corresponding eigenfunctions decay exponentially with the magnetic field and distance. We also prove that the integrated density of states is Lipschitz continuous away from the Landau energies. The proof relies on a Wegner estimate for the finite-area magnetic Hamiltonians with random potentials and exponential decay estimates for the finite-area Green's functions. The proof of the decay estimates for the Green's functions uses fundamental results from two-dimensional bond percolation theory.

Key-Words : Landau Hamiltonians, random operators, localization.

Number of figures : 4

August 1994
CPT-94/P.3061

anonymous ftp or gopher: cpt.univ-mrs.fr

*Unité Propre de Recherche 7061

¹ Département de Mathématiques, Université de Toulon et du Var - 83130 La Garde - France.

² Supported in part by CNRS.

³ Mathematics Department, University of Kentucky - Lexington KY 40506-0027 - USA.

⁴ Supported in part by NSF grants INT 90-15895 and DMS 93-07438.

1 Introduction

The existence of localized states for a two-dimensional gas of non-interacting electrons in a constant magnetic field is a main ingredient in various discussions and proofs of the integer quantum Hall effect (see e.g. [1], [2], [3], [4] and [7]). It is generally believed that localization occurs near the band edges for large magnetic fields and bounded, random potentials of arbitrary disorder. According to Halperin's argument [1], the localization length should diverge near the Landau levels. This is in contrast to the situation with no magnetic field. For two dimensional random systems, localization is expected to hold at all energies for arbitrary disorder and the eigenfunctions are expected to decay exponentially.

In this paper, we study the family H_ω of two-dimensional Landau Hamiltonians with Anderson-type potentials, having mean zero, on $L^2(\mathbb{R}^2)$. We prove that localization does occur in all energy intervals $I_n(B) \equiv [(2n+1)B + \mathcal{O}(B^{-1}), (2n+3)B - \mathcal{O}(B^{-1})]$, $n = 0, 1, 2, \dots$ at large magnetic field strengths B and for arbitrary disorder. Recall that $\sigma(H_\omega)$ is contained in bands about the Landau levels $E_n(B) \equiv (2n+1)B$, $n = 0, 1, 2, \dots$, of width $\|V\|_\infty$, independent of B . We follow the approach of [8] developed to study random Schrödinger operators on $L^2(\mathbb{R}^d)$. This work [8] extends to continuous systems the techniques of Howland [9], Simon-Wolff [10], von Dreifus-Klein [11], and Spencer [12].

For large magnetic fields, we justify a one Landau band approximation (of [13]) uniformly in n and obtain exponential decay estimates in x and B on the Green's function for finite-area Hamiltonians. The key to these estimates is showing that equipotential lines don't percolate with high probability. For potentials with zero averages, this holds at all energies except the Landau levels, which correspond to the critical percolation threshold. In addition to this restriction, there is a small region of energy of $\mathcal{O}(B^{-1})$ around each Landau level where small denominators in the interband perturbation expansion can't be controlled by our method. Although this is in agreement with an earlier conjecture of Laughlin [14], it remains an open question whether these small bands of energies about the Landau levels correspond to extended states.

In section 2 below, we describe the model and state the main results. We also give some elementary estimates needed later to justify the one Landau band approximation. In section 3, we prove Wegner estimates for the

quantum Hall Hamiltonian restricted to finite boxes. As a by-product, we obtain the Lipschitz continuity of the integrated density of states away from the Landau energies. The proofs of the exponential decay of the finite-area Green's function are given in section 4. The results of section 3 on the Wegner estimate and section 4 on the decay of the finite volume Hamiltonians are used in section 5, together with results of [8], to prove the main theorem. We prove some technical lemmas in the appendix.

We have recently learned of some related results on localization for the models studied here by J. Pulé [23] and by W. M. Wang [17].

Acknowledgments

We thank T. Hoffmann-Ostenhof and W. Thirring for their hospitality at the Erwin Schrödinger Institute in Vienna where most of this work was done. We thank R. Seiler and V. Jaksic for helpful discussions on the quantum Hall effect.

2 The Model and the Main Results

We consider a one-particle Hamiltonian which describes an electron in two-dimensions (x_1, x_2) subject to a constant magnetic field of strength $B > 0$ in the perpendicular x_3 -direction, and a random potential V_ω . The Hamiltonian H_ω has the form

$$H_\omega = (p - A)^2 + V_\omega, \quad (2.1)$$

on the Hilbert space $L^2(\mathbb{R}^2)$, where $p \equiv -i\nabla$, and the vector potential A is

$$A = \frac{B}{2}(x_2, -x_1), \quad (2.2)$$

so the magnetic field $B = \nabla \times A$ is in the x_3 -direction. The random potential V_ω is Anderson-like having the form

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) u(x - i). \quad (2.3)$$

We make the following assumptions on the single-site potential u and the coupling constants $\{\lambda_i(\omega)\}$.

(V1) $u \geq 0$, $u \in C^2$, $\text{supp } u \subset B(0, \frac{1}{\sqrt{2}})$, and $\exists C_0 > 0$ and $r_0 > 0$ s.t. $u|_{B(0, r_0)} > C_0$.

(V2) $\{\lambda_i(\omega)\}$ is an independent, identically distributed family of random variables with common distribution $g \in C^2([-M, M])$ for some $0 < M < \infty$, s.t. $\int \lambda g(\lambda) d\lambda = 0$ and $g(\lambda) > 0$ Lebesgue a.e. $\lambda \neq 0$.

We denote by $H_A \equiv (p - A)^2$, the Landau Hamiltonian. As is well-known, the spectrum of H_A consists of an increasing sequence $\{E_n(B)\}$ of eigenvalues, each of infinite multiplicity, given by

$$E_n(B) = (2n + 1)B, \quad n = 0, 1, 2, \dots \quad (2.4)$$

Note that $D(H_\omega) = D(H_A) \forall \omega \in \Omega$. We will call $E_n(B)$ the n^{th} Landau level and denote by P_n the projection onto the corresponding subspace. The orthogonal projection is denoted by $Q_n \equiv 1 - P_n$. Let $M_0 \equiv \sup_{x, \omega} \|V_\omega\| < \infty$.

Then, $\sigma(H_\omega) \subset \bigcup_{n \geq 0} \sigma_n$, where $\sigma_n \equiv [E_n(B) - M_0, E_n(B) + M_0]$, which we call the n^{th} Landau band. We show that $\sigma(H_\omega)$ is deterministic. The magnetic translations are defined for $a \in \mathbb{Z}^2$ by

$$U_a \equiv e^{-iBx \wedge a} e^{-ip \cdot a} \quad (2.5)$$

where $x \wedge a \equiv x_2 a_1 - x_1 a_2$. We then have

$$U_a H_\omega U_a^{-1} = H_{T_a \omega}, \quad (2.6)$$

where $T_a : \Omega \rightarrow \Omega$ is the \mathbb{Z}^2 -translation. Standard results (cf. [15]) show that H_ω is a \mathbb{Z}^2 -ergodic self-adjoint family of operators and consequently its spectrum is deterministic. Note that $\sigma(H_\omega)$ is not necessarily equal a.s. to $\bigcup_{n \geq 0} \sigma_n$. Provided some V_ω , $\omega \in \Omega_0$ and $|\Omega_0| > 0$, lifts the degeneracy of the Landau level, then ergodicity implies that the spectrum consists of bands each of which lies in some interval about $E_n(B)$, which might be strictly contained in σ_n .

Theorem 2.1 *Let H_ω be the family given in (2.1) with vector potential A satisfying (2.2), $B > 0$, and the random potential V_ω as in (2.3) and satisfying (V1)-(V2). Let*

$$I_n(B) \equiv [E_n(B) + \mathcal{O}(B^{-1}), E_{n+1}(B) - \mathcal{O}(B^{-1})].$$

There exists $B_0 \gg 0$ such that for $B > B_0$ and all $n = 0, 1, 2, \dots$,

$$\sigma(H_\omega) \cap I_n(B)$$

is pure point almost surely and the corresponding eigenfunctions decay exponentially. The integrated density of states is Lipschitz continuous away from $\sigma(H_A)$.

Let us make two remarks about the theorem. First, we note that the above theorem holds at arbitrary disorder. For large disorder, the techniques of [8] apply directly to show that, without the percolation estimates, $\sigma(H_\omega)$ is almost surely pure point in each Landau band. This regime, however, is of little interest as the quantum Hall conductivity vanishes in this case. Secondly, we show, in fact, that the localization length, for energies near the band edges as in Theorem 2.1, is a decreasing function of the field strength B so that the wave functions are strongly localized. We also show that the localization length increases as the energy approaches the Landau levels. The precise manner in which this occurs follows from Proposition 5.1 and Theorem 5.2. However, our method fails to give an estimate of the power law divergence of the localization length near the Landau level.

As is clear from the Wegner estimate, Theorem 3.1, our method fails to give information about the integrated density of states at the Landau energies. However, we can improve the result if we make a stronger hypothesis in (V1) on $\text{supp } u$.

Corollary 2.1 *If, in addition to the hypothesis of Theorem 2.1, we have $u \geq C_0 \chi_{\Lambda_1(0)}$, $C_0 > 0$, then the integrated density of states is Lipschitz continuous.*

If the hypothesis of Corollary 2.1 does not hold, then a large portion of configuration space is unaffected by the potential. It is not, therefore, surprising that there is a discontinuity in the integrated density of states at the Landau energies as there is for the Landau Hamiltonian. A phenomenon of this type has been observed by Brezin et. al. [6] for a Poisson distribution of impurities at low energy. So we do not expect that the IDS is Lipschitz continuous at the Landau energies without a condition of the support of u which implies that the zero set of V_ω is in some sense “small”.

We mention that W.M. Wang [16] has obtained an asymptotic expansion in the semi-classical limit for the density of states at large magnetic field strengths away from the Landau levels, partially justifying the one-band approximation.

We conclude this section with some simple observations on the Landau projections P_n .

The projection P_n on the n^{th} Landau level of H_A has a kernel given by

$$P_n(x, y) = B e^{-\frac{iB}{2} x \wedge y} p_n \left(B^{\frac{1}{2}}(x - y) \right), \quad (2.7)$$

where $p_n(x)$ is of the form

$$p_n(x) = \left\{ n^{\text{th}} \text{ degree polynomial in } x \right\} e^{-\frac{|x|^2}{2}}, \quad (2.8)$$

and independent of B . We will make repeated use of the following elementary lemma, the proof of which follows by direct calculation using the kernel (2.7)-(2.8).

Lemma 2.1 *Let χ_1, χ_2 be functions of disjoint, compact support with $|\chi_i| \leq 1$ and let $\delta \equiv \text{dist}(\text{supp } \chi_1, \text{supp } \chi_2) > 0$. Then,*

- (1) $\|\chi_1 P_n \chi_1\|_1 \leq C_n B |\text{supp } \chi_1|^2$;
- (2) $\|\chi_1 P_n \chi_1\|_{HS} \leq C_n B |\text{supp } \chi_1|$;
- (3) $\|\chi_1 P_n \chi_2\|_{HS} \leq C_n B |p_n(B^{\frac{1}{2}}\delta)|^{\frac{1}{2}} \{|\text{supp } \chi_1| |\text{supp } \chi_2|\}^{\frac{1}{2}}$;
- (4) $\|\chi_1 P_n\|_{HS} \leq C_n B$,

where C_n varies from line to line and depends only on χ_i and n , and HS denotes the Hilbert Schmidt norm. Note that $|p_n(B^{\frac{1}{2}}\delta)|^{\frac{1}{2}} \leq C_0 e^{-\epsilon B}$, for any $\epsilon > 0$ and B large enough.

3 Wegner Estimate

We define local Hamiltonians as relatively compact perturbations of the Landau Hamiltonian $H_A = (p - A)^2$, as defined in section 2. Let $\Lambda \subset \mathbb{R}^2$ denote

an open connected region in \mathbb{R}^2 . We let $\Lambda_\ell(x)$ denote a square of side ℓ centered at $x \in \mathbb{R}^2$,

$$\Lambda_\ell(x) \equiv \left\{ y \in \mathbb{R}^2 \mid |x_i - y_i| < \ell, \ i = 1, 2 \right\}.$$

Given $\Lambda \subset \mathbb{R}^2$, the local potential V_Λ is defined as follows. Freeze all $\lambda_j(\omega) \in \mathbb{Z}^2 \cap (\mathbb{R}^2 \setminus \Lambda)$ and consider \tilde{V} so obtained. This potential depends on the external, fixed coupling constants and on all $\lambda_i(\omega) \in \mathbb{Z}^2 \cap \Lambda$. We define $V_\Lambda \equiv \tilde{V}|_\Lambda$ and define $H_\Lambda \equiv H_A + V_\Lambda$ on $L^2(\mathbb{R}^2)$. These Hamiltonians are not independent of the external configurations but we will prove estimates uniform in the external random variables. We will use the conditional probability law

$$\overline{P}(A \cap B) \leq \overline{P}(A)\overline{P}(B), \quad (3.1)$$

where \overline{P} is the probability conditioned on the external variables and A & B are any two events in Λ . Note that $\sigma_{\text{ess}}(H_\Lambda) = \sigma_{\text{ess}}(H_A)$, since V_Λ is relatively compact.

We prove the following theorem.

Theorem 3.1 *$\exists B_0 > 0$ and a constant $C_W > 0$ such that for all $B > B_0$ and for any $E \notin \sigma(H_A)$,*

$$\mathbb{P}\{\text{dist}(\sigma(H_\Lambda), E) < \delta\} \leq C_W [\text{dist}(\sigma(H_\Lambda), E)]^{-2} \|g\|_\infty \delta B |\Lambda|.$$

This theorem will follow from the properties of the spectral projectors for H_A and a spectral averaging theorem. Since H_Λ depends analytically on the coupling constants λ_i , we need only a simple version which we state without proof (cf. [8], [18], [19]).

Lemma 3.1 *Let $H_\lambda \equiv H_0 + \lambda u$, $\lambda \in \mathbb{R}$, be a self-adjoint family on $D(H_0)$ with $C_0 D^2 \leq u \leq M < \infty$. Let $E_\lambda(\cdot)$ be the spectral family for H_λ . For any $h \geq 0$, $\text{supp } h$ compact, and $h \in L^\infty(\mathbb{R})$, and for any $L \subset \mathbb{R}$ measurable, we have*

$$\left\| \int_{\mathbb{R}} h(\lambda) dE_\lambda(L) d\lambda \right\| \leq C_0^{-1} |L| \|h\|_\infty.$$

For simplicity, we will work with the case $n = 0$, the first Landau band, although the calculation is uniform in n . As can be easily checked, the calculations depend only on the difference between the energy E that we are

considering and the nearest Landau energy $E_n(B)$ and hence is independent of n . To begin the proof of Theorem 3.1, we need a simple estimate. Let Δ be an interval in the first Landau band σ_0 . Let E_Δ be the spectral projector for H_Λ associated with Δ .

Lemma 3.2 $\|E_\Delta Q_0 E_\Delta\| \leq d_\Delta^{-\frac{1}{2}} (1 - (2d_\Delta)^{-1}|\Delta|)^{-\frac{1}{2}} M^{\frac{1}{2}}$, where $d_\Delta \equiv \text{dist}(\sigma(H_A) \setminus \{B\}, \Delta) = \mathcal{O}(B)$.

Proof

Let $E_m \in \Delta$ be the center of the interval. We then can write

$$\begin{aligned} E_\Delta Q_0 E_\Delta &\leq \text{dist}(\sigma(H_A) \setminus \{B\}, \Delta)^{-1} (E_\Delta (H_A - E_m) Q_0 E_\Delta) \\ &\leq d_\Delta^{-1} \{E_\Delta (H_\Lambda - E_m) Q_0 E_\Delta + E_\Delta V_\Lambda Q_0 E_\Delta\}. \end{aligned}$$

This implies that

$$\|E_\Delta Q_0 E_\Delta\| \leq d_\Delta^{-1} \left\{ \frac{|\Delta|}{2} \|E_\Delta Q_0 E_\Delta\| + M_0 \|Q_0 E_\Delta\| \right\}.$$

Since $d_\Delta = \mathcal{O}(B)$, it is clear that for all B sufficiently large $(2d_\Delta)^{-1}|\Delta| \ll 1$, so

$$\|E_\Delta Q_0 E_\Delta\| \leq d_\Delta^{-1} \left(1 - (2d_\Delta)^{-1}|\Delta|\right)^{-1} M_0 \|E_\Delta Q_0 E_\Delta\|^{\frac{1}{2}},$$

and the result follows. ■

Note that as $d_\Delta = \mathcal{O}(B)$, we obtain

$$\|E_\Delta Q_0 E_\Delta\| = \mathcal{O}\left(B^{-\frac{1}{2}}\right). \quad (3.2)$$

Proof of Theorem 3.1

We can assume without loss of generality that the closest point in $\sigma(H_A)$ to E is $E_0(B) = B$. All the calculations below hold for any band. Let $\Delta \subset \sigma_0 \setminus \{E_0(B)\}$ be a connected interval containing E and let E_Δ be the spectral projection for H_Λ and Δ . Recall from Chebyshev's inequality that

$$\mathbb{P}_\Lambda\{\text{dist}(\sigma(H_\Lambda), E) < \eta\} \leq \mathbb{E}_\Lambda(\text{Tr} E_\Delta), \quad (3.3)$$

where \mathbb{P}_Λ and \mathbb{E}_Λ denote the probability and expectation with respect to the variables in $\Lambda \cap \mathbb{Z}^2$ and Tr denotes the trace on $L^2(\mathbb{R}^2)$. We first note that

$$\text{Tr} E_\Delta \leq 2\text{Tr}(P_0 E_\Delta P_0). \quad (3.4)$$

This follows from the identity

$$\text{Tr} E_\Delta = \text{Tr} E_\Delta P_0 E_\Delta + \text{Tr} E_\Delta Q_0 E_\Delta,$$

and the bound

$$\text{Tr} E_\Delta Q_0 E_\Delta \leq \|E_\Delta Q_0 E_\Delta\| (\text{Tr} E_\Delta),$$

since $E_\Delta Q_0 E_\Delta \geq 0$. Now by Lemma 3.2, $\|E_\Delta Q_0 E_\Delta\| = \mathcal{O}(B^{-\frac{1}{2}})$ so (3.4) follows for all B sufficiently large. Let us now suppose $\inf \Delta > B$ for definiteness. From (3.4), and positivity we obtain

$$\begin{aligned} \text{Tr} E_\Delta P_0 E_\Delta &\leq \text{Tr}(E_\Delta(H_\Lambda - B)P_0(H_\Lambda - B)E_\Delta) \cdot \text{dist}(\Delta, B)^{-2} \\ &\leq \text{Tr}(P_0 V_\Lambda E_\Delta V_\Lambda P_0) \cdot \text{dist}(\Delta, B)^{-2}. \end{aligned} \quad (3.5)$$

Writing $V_\Lambda = \sum_i \lambda_i u_i$ for short, the trace in (3.5) is

$$\sum_{i,j} \lambda_i \lambda_j \text{Tr}(P_0 u_i E_\Delta u_j P_0), \quad (3.6)$$

where $i, j \in \Lambda \cap \mathbb{Z}^2$. Defining $A^{ij} \equiv u_i^{\frac{1}{2}} A u_j^{\frac{1}{2}}$ for any $A \in B(\mathcal{H})$, we have from (3.6),

$$\sum_{i,j} \lambda_i \lambda_j \text{Tr}(P_0^{ij} E_\Delta^{ij}). \quad (3.7)$$

We must estimate

$$\begin{aligned} \mathbb{E}_\Lambda \left(\sum_{i,j} \lambda_i \lambda_j \text{Tr}(P_0^{ij} E_\Delta^{ij}) \right) &\leq \sum_{i,j} \mathbb{E}_\Lambda (|\lambda_i \lambda_j| |\text{Tr}(P_0^{ij} E_\Delta^{ij})|) \\ &\leq \frac{1}{2} M^2 \sum_{i,j} \|P_0^{ij}\|_1 \mathbb{E}_\Lambda \{ \|E_\Delta^{ii}\| + \|E_\Delta^{jj}\| \}. \end{aligned} \quad (3.8)$$

Since $E_\Delta^{ii} \equiv u_i^{\frac{1}{2}} E_\Delta u_i^{\frac{1}{2}} \geq 0$ and self-adjoint, we have

$$\begin{aligned} \mathbb{E}_\Lambda (\|E_\Delta^{ii}\|) &\leq \sup_{\psi, \|\psi\|=1} \{ \mathbb{E}_\Lambda (\langle \psi, E_\Delta^{ii} \psi \rangle) \} \\ &\leq C_0^{-1} \|g\|_\infty |\Delta|, \end{aligned}$$

by Lemma 3.1. Consequently, (3.8) is bounded above by

$$\frac{1}{2} C_0^{-1} M^2 \|g\|_\infty |\Delta| \sum_{i,j} \|P_0^{ij}\|_1. \quad (3.9)$$

To evaluate the trace norm, we first note that for $i = j$, $P_0^{ii} > 0$ so by Lemma 2.1,

$$\sum_{i=j} \|P_0^{ij}\|_1 = C_0 B |\text{supp } u|^2 |\Lambda|. \quad (3.10)$$

Next, suppose $u_i u_j \neq 0$, $i \neq j$. Let χ_{ij} be the characteristic function on $\text{supp } u_i u_j$. Then, if $i \cap j$ denotes the set of such pairs,

$$\sum_{i \cap j} \|u_i^{\frac{1}{2}} P_0 u_j^{\frac{1}{2}}\|_1 \leq \sum_{i \cap j} \|\chi_{ij} P_0 \chi_{ij}\|_1 \leq C_1 B |\Lambda| |\text{supp } u|^2, \quad (3.11)$$

where we used $\sup u_i \leq 1$. Finally, for $u_i u_j = 0$, let $\{\chi_\ell^2\}$ be a partition of unity covering Λ so that $\chi_\ell |\text{supp } u_\ell| = 1$ and $\chi_\ell \chi_n = 0$, $\ell \neq n$. Using the inequality

$$\|AB\|_1 \leq \|A\|_{HS} \|B\|_{HS}, \quad (3.12)$$

we obtain

$$\|u_j^{\frac{1}{2}} P_0 u_j^{\frac{1}{2}}\| \leq \sum_{\ell} \|u_j^{\frac{1}{2}} P_0 \chi_\ell\|_{HS} \|\chi_\ell P_0 u_j^{\frac{1}{2}}\|_{HS}. \quad (3.13)$$

As in Lemma 2.1 with $\delta \equiv |i - \ell| - 2r_u$, we easily compute

$$\|u_i^{\frac{1}{2}} P_0 \chi_\ell\|_{HS} \leq C_2 B e^{-B(|i-\ell|-2r_u)}, \quad (3.14)$$

where $0 < r_u < \frac{1}{\sqrt{2}}$ is the radius of $\text{supp } u_i$ (and χ_ℓ) (see (V1)). Summing over $\{ij\}'$, the set of pairs with $u_i u_j = 0$, we get from (3.13)-(3.14),

$$\begin{aligned} \sum_{\{ij\}'} \|P_0^{ij}\|_1 &\leq C_3 B^2 \sum_{\{ij\}' \ell} e^{-B(|i-\ell|-2r_u)} e^{-B(|j-\ell|-2r_u)} \\ &\leq C_4 B^2 |\Lambda| e^{-\delta B}, \end{aligned} \quad (3.15)$$

for same $\delta > 0$. Combining (3.10), (3.11) and (3.15) in (3.9) we obtain an upper bound for all B large enough,

$$\mathbb{E}(\text{Tr } E_\Delta) \leq C_W \|g\|_\infty |\Delta| B |\Lambda|,$$

where C_W depends on M^2 , C_0^{-1} , and $|\text{supp } u|^2$. This proves the theorem. ■

The estimate of Theorem 3.1 suffices to prove the Lipschitz continuity of the integrated density of states away from the Landau levels, as stated in Theorem 2.1. With regard to Corollary 2.1, let us show how the additional

hypothesis on $\text{supp } u$ allows the improvement. For $M_0 \equiv \|V_\omega\|_\infty$ as in section 2, define

$$H_0 = H_A + 2M_0(1 - \chi_\Lambda),$$

and the finite-area Hamiltonian by

$$H_\Lambda = H_0 + V_\Lambda.$$

Beginning with (3.4), we have for $\Delta \subset \sigma_0$ and $E_m \equiv \text{center of } \Delta$,

$$\begin{aligned} \text{Tr} E_\Delta &\leq 2\text{Tr} \{E_\Delta(H_A + 2M_0 - E_m)P_0(B + 2M_0 - E_m)^{-1}\} \\ &\leq 2(B + 2M_0 - E_m)^{-1} \{ \text{Tr} E_\Delta(H_\Lambda - E_m)P_0 \\ &\quad + \text{Tr} E_\Delta(2M_0\chi_\Lambda - V_\Lambda)P_0 \}. \end{aligned}$$

Since $2M_0\chi_\Lambda - V_\Lambda > M_0\chi_\Lambda$ and $\|(H_\Lambda - E_m)E_\Delta\| \leq \frac{|\Delta|}{2}$, we obtain

$$\text{Tr} E_\Delta \leq 2(B + 2M_0 - E_m)^{-1} \left\{ \frac{|\Delta|}{2} \text{Tr} P_0 E_\Delta + M_0 \text{Tr} E_\Delta \chi_\Lambda P_0 \right\}.$$

As $(B + 2M_0 - E_m)^{-1}|\Delta| < \frac{|\Delta|}{M_0} \ll 1$, we arrive at

$$\text{Tr} E_\Delta \leq 4M_0 C_1^{-1} (B + 2M_0 - E_m)^{-1} \left\{ \sum_{i \in \Lambda \cap \mathbb{Z}^2} \text{Tr}(E_\Delta u_i P_0) \right\}.$$

Here we used the fact that $\sum_{i \in \mathbb{Z}^2 \cap \Lambda} u_i \geq C_1 \chi_\Lambda$. The remaining steps are the same as above. In light of this calculation one might speculate that the singularity at the Landau energies of the IDS is due to the existence of large regions where there is no potential. Indeed, numerical studies on the Poisson model [6] seem to also support this idea.

4 Percolation Theory and Decay Estimates

In this section, we prove the technical estimates required to justify the one-Landau band approximation. We consider for simplicity the first Landau band $\sigma_0 \equiv [B - M_0, B + M_0]$, but all other bands can be analysed using the same techniques. The results are uniform in the band index n . Formally, if

one neglects the band interaction, the effective Hamiltonian for an electron at energy E is $E = B + V(x)$. Consequently, in this approximation, the electron motion is along equipotential lines $V(x) + B - E = 0$. Since V is random, it is natural to estimate the probability that these equipotential lines percolate through a given box. If not, the electron will remain confined to bounded regions. One can expect that the interband interaction will not change this picture. We will do this in the second part of this section by showing that the Green's function decays exponentially in x and B through regions where $|V(x) + B - E| > a > 0$. The first part of this section is devoted to reformulating our problem as a problem in bond percolation.

4.1 Percolation Estimates

We first show that in annular regions between boxes of side ℓ and $\ell/3$ there exist closed, connected ribbons where the condition $|V(x) + B - E| > a > 0$ is satisfied, provided $E \neq B$, with a probability which converges exponentially fast to 1 as ℓ tends to infinity. Obviously, the existence of such a ribbon is equivalent to the impossibility for equipotential lines at energy E to percolate from the center of the box to its boundary. Although this is a classical matter, let us recall how one can formulate the above condition in terms of two-dimensional bond percolation.

Recall that $V_\omega(x) = \sum_{i \in \mathbb{Z}^2} \lambda_i(\omega) u(x - i)$, where the single-site potential $u \geq 0$ and has support inside a ball of radius $r_u < \frac{1}{\sqrt{2}}$. We define r_u to be the smallest radius such that $\text{supp } u \subset B(0, r_u)$. Consider a new square lattice $\Gamma \equiv e^{i\pi/4} \sqrt{2} \mathbb{Z}^2$. The midpoint of each bond of Γ is a site of \mathbb{Z}^2 (see Figure 1). We will denote by b_j the bond of Γ having $j \in \mathbb{Z}^2$ as its midpoint. For definiteness, we assume $E \in (B, B + M_0)$. The other energy interval can be treated similarly.

Definition 4.1 *The bond b_j of Γ is **occupied** if $\lambda_j(\omega) < \frac{E-B}{2}$. The probability $\mathbb{P} \left\{ \lambda_j(\omega) < \frac{E-B}{2} \right\} \equiv p$ is the probability that b_j is occupied (p is independent of j by the iid assumption).*

Let us assume that the bond b_j is occupied and consider (see Figure 2),

$$\mathcal{R}_j \equiv \left\{ x \mid \text{dist}(x, b_j) < \frac{1}{\sqrt{2}} - r_u \equiv r_1 \right\}. \quad (4.1)$$

Obviously, \mathcal{R}_j does not intersect the support of the other single-site potentials centered on $\mathbb{Z}^2 \setminus \{j\}$ so that $V(x) = \lambda_j(\omega)u(x-j) \forall x \in \mathcal{R}_j$. Then, if b_j is occupied, one has $V(x) < \frac{E-B}{2} \forall x \in \mathcal{R}_j$ (recall that $\frac{E-B}{2} > 0$). We now assume that there is a closed circuit of occupied bonds $\mathcal{C} \equiv \bigcup_{j \in \gamma} b_j$, $\gamma \subset \mathbb{Z}^2$ (i.e. a connected union occupied bonds). We call $\mathcal{R} \equiv \bigcup_{j \in \gamma} \mathcal{R}_j$ the closed *ribbon* associated with \mathcal{C} . For all $x \in \mathcal{R}$, we have $V(x) < \frac{E-B}{2}$. If we take $a \equiv \frac{E-B}{2}$, then

$$V(x) + B - E < -a \forall x \in \mathcal{R}. \quad (4.2)$$

The existence of a closed ribbon \mathcal{R} so that V satisfies condition (4.2) is a consequence of the existence of a closed circuit \mathcal{C} in Γ of occupied bonds. In order to estimate the probability that \mathcal{C} exists, we use some standard results of percolation theory (see, e.g. [20] and [21]) which we now summarize.

Let \mathbb{Z}^2 be the square lattice (the length of the side plays no role in the calculations). A *bond* (edge) of \mathbb{Z}^2 is said to be *occupied* with probability p , $0 \leq p \leq 1$, and *empty* with probability $1 - p$. We are interested in the case when the bonds are independent (called Bernoulli bond percolation). The critical percolation probability p_c is defined as follows. Let $P_\infty(p)$ be the probability that the origin belongs to an infinite (connected) cluster of occupied bonds. Then, we define

$$p_c \equiv \inf\{p | P_\infty(p) > 0\}.$$

For 2-dimensional Bernoulli bond percolation, $p_c = \frac{1}{2}$. Hence if $p > p_c$, occupied bonds percolate ; that is, we can find a connected cluster of occupied bonds running off to infinity with non-zero probability.

Of importance for us are the results concerning the existence of closed circuits of occupied bonds. Let $r_{n,\ell}$ be a rectangle in \mathbb{Z}^2 of width ℓ and length $n\ell$. Let $R_{n,\ell}$ be the probability that there is a crossing of $r_{n,\ell}$, the long way, by a connected path of occupied bonds. A basic result is

Theorem 4.1 *For $p > p_c$, $R_{n,\ell} \geq 1 - C_0 n \ell e^{-m(1-p)\ell}$, for some constant C_0 .*

The exponential factor $m(q)$ is strictly positive for $q < p_c$ and $m(q) \searrow 0$ as $q \nearrow p_c$. This factor measures the probability that the origin 0 is connected to $x \in \mathbb{Z}^2$ by a path of occupied bonds

$$P_{0x}(p) \leq e^{-m(p)|x|}.$$

Let us write r_ℓ for $r_{1,\ell}$, the box of side ℓ . An annular region between two concentric boxes is denoted by $a_\ell \equiv r_{3\ell} \setminus r_\ell$. A closed circuit of occupied bonds in a_ℓ is a connected path of occupied bonds lying entirely within a_ℓ . Using Theorem 4.1 and the *FGK* inequality, one can compute the probability A_ℓ of a closed circuit of occupied bonds in a_ℓ for $p > p_c$.

Theorem 4.2 *For any $p \in [0, 1]$, $A_\ell \geq [R_{3,\ell}(p)]^4$. In particular, if $p > p_c$, $\exists 0 < C_0 < \infty$ as in Theorem 4.1, such that*

$$A_\ell \geq 1 - 12C_0 e^{-m(1-p)\ell}. \quad (4.3)$$

We now apply these results to our situation as follows. On the lattice Γ defined above, the probability that any bond is occupied is given by

$$p = \int_{-M}^a g(\lambda) d\lambda,$$

so, under our assumptions on the density g , if $a > 0$ then $p > p_c = \frac{1}{2}$, and we are above the critical percolation threshold $p_c = \frac{1}{2}$. Note that when $E = B$, $a = 0$ so $p = \frac{1}{2} = p_c$, the critical probability. It follows from Theorem 4.2 that any annular region $a_\ell \equiv r_{3\ell} \setminus r_\ell$ in Γ of diameter $\sqrt{2}\ell \equiv \frac{1}{2}(3\sqrt{2}\ell - \sqrt{2}\ell)$ and sides parallel to the bonds of Γ (see Figure 3) contains a closed circuit of occupied bonds with probability given by (4.3). By the argument above, there is a ribbon \mathcal{R} associated with \mathcal{C} in a_ℓ whose properties we summarize in the next proposition.

Proposition 4.1 *Assume (V1) and (V2). Let $r_u \equiv \frac{1}{2} \text{diam}(\text{supp } u)$, $\ell > \sqrt{2}$, $E \in \sigma_0 \setminus \{B\}$, and $a > 0$. Then for $m(q)$ and C_0 as in Theorem 4.2, \exists a ribbon \mathcal{R} satisfying*

$$\text{diam} \mathcal{R} \geq 2 \left(\frac{1}{\sqrt{2}} - r_u \right); \quad (4.4)$$

$$\text{dist}(\mathcal{R}, \partial r_{3\ell}), \text{dist}(\mathcal{R}, \partial r_\ell) \geq \frac{1}{\sqrt{2}} + r_u; \quad (4.5)$$

$$\mathcal{R} \subset a_\ell,$$

and s.t.

$$V(x) + B - E < -a, \quad \forall x \in \mathcal{R}, \quad (4.6)$$

with a probability larger than

$$1 - 12C_0 e^{-m(1-p)\ell}, \quad (4.7)$$

where

$$p \equiv \int_{-M}^a g(\lambda) d\lambda. \quad (4.8)$$

4.2 Decay Estimates

The effective one Landau band Hamiltonian $B + V$ localizes electrons at energies E where the equipotential lines $E = V(x) + B$ don't percolate to infinity. The effect of the interband interaction is to induce some tunneling through the ‘‘Classically Forbidden’’ ribbon \mathcal{R} of Proposition 4.1. As a consequence, instead of localization in the compact subsets of \mathbb{R}^2 bounded by \mathcal{R} , one expects exponential decay of the Green's function in x and B across such ribbons \mathcal{R} . Such an estimate is the starting point of the inductive, multi-scale analysis detailed in section 5. By the geometric resolvent equation, we show there that it suffices to consider the following ideal situation, where for some $a > 0$,

$$V(x) + B - E < -a, \quad \forall x \in \mathbb{R}^2, \quad (4.9)$$

or, alternately,

$$V(x) + B - E > a, \quad \forall x \in \mathbb{R}^2. \quad (4.10)$$

A condition such as (4.9) with $E > B$ is satisfied, with a probability given in Proposition 4.1, by a smoothing (see section 5) of the potential $V_{\mathcal{R}}$ defined as

$$V_{\mathcal{R}}(x) = \begin{cases} V(x) & x \in \mathcal{R} \\ 0 & x \in \mathbb{R}^2 \setminus \mathcal{R}. \end{cases} \quad (4.11)$$

Here we obtain decay estimates on

$$H = H_A + V$$

with V having compact support with non-empty interior and satisfying (4.9) or (4.10).

Let \mathcal{O} be an open, bounded, connected set in \mathbb{R}^2 with smooth boundary and define $\rho(x) = \text{dist}(x, \mathcal{O})$. Let $\eta \in C_0^\infty(\mathbb{R}^2)$ with $\eta > 0$ and $\text{supp } \eta \subset B_1(0)$. For any $\epsilon > 0$, define $\eta_\epsilon(x) = \eta(x/\epsilon)$. We consider the smoothed

distance function $\rho_\epsilon(x) \equiv (\eta_\epsilon \star \rho)(x)$; $\text{supp } \rho_\epsilon \subset \mathbb{R}^2 \setminus \{x | \text{dist}(x, \mathcal{O}^c) < \epsilon\}$. We fix $\epsilon > 0$ small and write ρ for ρ_ϵ below for simplicity. We have $\|\nabla \rho\|_\infty < C_0/\epsilon$ and $\|\Delta \rho\|_\infty < C_1/\epsilon^2$, for constants $C_0, C_1 > 0$ depending only on η and \mathcal{O} . This ϵ will play no role in the analysis below and, consequently, we absorb it into the constants C_0 and C_1 . We consider one-parameter families of operators defined for $\alpha \in \mathbb{R}$ as

$$H_A(\alpha) \equiv e^{i\alpha\rho} H_A e^{-i\alpha\rho}; \quad (4.12)$$

$$H(\alpha) \equiv H_A(\alpha) + V; \quad (4.13)$$

$$P(\alpha) \equiv e^{i\alpha\rho} P e^{-i\alpha\rho}, \text{ etc.}, \quad (4.14)$$

and similarly for the local Hamiltonian $H_\Lambda(\alpha) \equiv H_A(\alpha) + V_\Lambda$. Here, we write P for the projector P_0 . For $\alpha \in \mathbb{R}$, these families are unitarity equivalent with the $\alpha = 0$ operator.

Lemma 4.1. *The family $H(\alpha)$ (and similarly for $H_\Lambda(\alpha)$), $\alpha \in \mathbb{R}$, has an analytic continuation into the strip*

$$S \equiv \{\alpha \in \mathbb{C} | \text{Im } \alpha < \eta_\rho B^{1/2},\} \quad (4.15)$$

as a type A analytic family with domain $D(H)$. The positive constant η_ρ depends only on the distance function ρ . Furthermore, in this strip S , one has $P(\alpha)^2 = P(\alpha)$ and for some constant C_1 independent of α ,

$$\|P(\alpha)\| < C_1 \quad (4.16)$$

and

$$\|Q(\alpha)(H_A(\alpha) - z)^{-1}\| < C_1 B^{-1}, \text{ if } \text{dist}(z, B) \leq B. \quad (4.17)$$

Proof. For $\alpha \in \mathbb{R}$, one has

$$\begin{aligned} H_A(\alpha) &= (-i\nabla - \alpha\nabla\rho - A)^2 \\ &= H_A - \alpha[\nabla\rho \cdot (p - A) + (p - A) \cdot \nabla\rho] + \alpha^2|\nabla\rho|^2 \\ &= H_A + \alpha^2|\nabla\rho|^2 + i\alpha\Delta\rho - 2\alpha\nabla\rho \cdot (p - A). \end{aligned} \quad (4.18)$$

By a standard unitary equivalence argument, it suffices to show that

$$\{\alpha^2|\nabla\rho|^2 + i\alpha\Delta\rho - 2\alpha\nabla\rho \cdot (p - A)\}(H_A - z)^{-1},$$

has norm less than 1 for some $z \notin \sigma(H_A)$ and α purely imaginary (cf [22]). For later purposes, we choose $z \in C(B) \equiv \{z \mid |z - B| = B\}$, circle of radius B centered at B . Since $|\nabla \rho|^2$ and $\Delta \rho$ are bounded, we can choose η'_ρ such that

$$\|(\alpha^2 |\nabla \rho|^2 + i\alpha \Delta \rho)(H_A - z)^{-1}\| < 1/2,$$

for $|\alpha| < \eta'_\rho B^{1/2}$ and for all $z \in C(B)$. Hence, it is enough to show that $\forall |\alpha| < \eta''_\rho B^{1/2}$, for a possibly smaller constant η''_ρ ,

$$2|\alpha| \|\nabla \rho \cdot (p - A)(H_A - z)^{-1}\| < 1/2, \quad (4.19)$$

for some $z \in C(B)$. Since $\|\nabla \rho\|_\infty < C_0$, we easily find that $\|\nabla \rho \cdot (p - A)(H_A - z)^{-1}\| < C_1 B^{-1/2}$. This implies (4.19) for all B sufficiently large. We take η_ρ to be the smallest of these two constants. From these estimates and (4.18) for $\alpha \in S$ we have that for B large enough,

$$\|(H_A(\alpha) - z)^{-1}\| < C_2 B^{-1}, \quad z \in C(B), \quad (4.20)$$

for some constant C_2 uniform in $\alpha \in S$ and $z \in C(B)$. Next, note that the eigenfunctions of H_A are analytic vectors for the family $e^{i\alpha\rho}$, $\alpha \in S$. It is a consequence of this, the analyticity of H_A , and the eigenvalue equation, that $\sigma(H_A(\alpha))$ is independent of α , $\alpha \in S$. The family $P(\alpha)$, $\alpha \in \mathbb{R}$, has an analytic continuation in S given by the contour integral

$$P(\alpha) = \frac{-1}{2\pi i} \int_{C(B)} (H_A(\alpha) - z)^{-1} dz. \quad (4.21)$$

The boundedness of $P(\alpha)$ follows from (4.20) and (4.21). The idempotent property of $P(\alpha)$ follows from this analyticity and the identity $P(\alpha)^2 = P(\alpha)$, which holds for real α . Furthermore, the function $\alpha \in S \rightarrow Q(\alpha)(H_A(\alpha) - z)^{-1}$ is holomorphic on and inside $C(B)$. By the maximum modulus principle, it follows that

$$\|Q(\alpha)(H_A(\alpha) - z)^{-1}\| \leq \sup_{z \in C(B)} \|Q(\alpha)(H_A(\alpha) - z)^{-1}\|, \quad (4.22)$$

and the bound (4.17) follows from this, (4.16) and (4.20). \square

We next prove the main estimate of this section.

Theorem 4.3. *Assume that (V, E, B) satisfy (4.9) or (4.10) for some $a > 0$ and $E \in \sigma_0 \setminus \{B\}$. Furthermore, assume that $\text{supp } V$ is compact with non-empty interior. There exists constants $C_2 \leq \eta_\rho$, C_3 , and B_1 , depending*

only on $M_0 \equiv \|V\|_\infty, \|\nabla\rho\|_\infty$, and $\|\nabla V\|_\infty$, such that if we define $\gamma \equiv C_2 \min\{B^{1/2}, aB\}$, and u is a solution of

$$(H_A + V - z)u = v, \quad z \equiv E + i\epsilon, \quad \epsilon > 0, \quad E > 0, \quad (4.23)$$

for some $v \in D(e^{\gamma\rho})$, then for $B > B_1$, $\forall \alpha \in \mathbb{C}$, $|\operatorname{Im} \alpha| < \gamma$, we have

$$u \in D(e^{i\rho\alpha}), \quad (4.24)$$

$$\|e^{\alpha\rho}Pu\| \leq C_3 a^{-1} \|e^{\alpha\rho}v\|, \quad (4.25)$$

and

$$\|e^{\alpha\rho}Qu\| \leq C_3 B^{-1} \|e^{\alpha\rho}v\|. \quad (4.26)$$

Proof. Let $v(\alpha) = e^{i\alpha\rho}v$, so that $v(\alpha)$ is analytic in the strip $|\operatorname{Im}(\alpha)| < \gamma$. Let $u(\alpha) = e^{i\alpha\rho}u$, $\alpha \in \mathbb{R}$, i.e.,

$$u(\alpha) = (H(\alpha) - z)^{-1}v(\alpha) \equiv ((H - z)^{-1}v)(\alpha), \quad (4.27)$$

with $H \equiv H_A + V$, as above. Since V is H_A -compact by assumption, H has point spectrum, and according to Lemma 4.1 and standard arguments, $H(\alpha)$ has real spectrum independent of α in the strip S defined in (4.15). Then, $u(\alpha)$ has an analytic continuation in $|\operatorname{Im}(\alpha)| < \gamma$, which proves (4.24).

Projecting the equation (4.23) for u along $P(\alpha)$ gives

$$(B + V - z)(Pu)(\alpha) = (Pv)(\alpha) + ([QVP - PVQ]u)(\alpha). \quad (4.28)$$

Taking the scalar product of (4.28) with $(Pu)(\alpha)$ results in the inequality,

$$\begin{aligned} a\|(Pu)(\alpha)\|^2 &\leq \|(Pu)(\alpha)\| \|(Pv)(\alpha)\| \\ &\quad + \{ \|(P^*Q)(\alpha)\| \|(QVP)(\alpha)\| \} \|(Pu)(\alpha)\|^2 \\ &\quad + \|(PVQ)(\alpha)\| \|(Pu)(\alpha)\| \|(Qu)(\alpha)\|. \end{aligned} \quad (4.29)$$

In the appendix, we prove that for B large enough,

$$\|(QVP)(\alpha)\| \leq C_4 B^{-1/2},$$

and

$$|(P^*Q)(\alpha)| \leq C_5 |Im\alpha| B^{-1/2}.$$

With these estimates, we obtain from (4.29,)

$$\begin{aligned} (a - C_6\gamma B^{-1})\|(Pu)(\alpha)\|^2 &\leq \|(Pu)(\alpha)\| \|(Pv)(\alpha)\| \\ &\quad + C_7 B^{-1/2} \|(Pu)(\alpha)\| \|(Qu)(\alpha)\|, \end{aligned} \quad (4.30)$$

where the constants C_6 and C_7 depend only on $\|V\|_\infty$, $\|\nabla V\|_\infty$, and $\|\nabla\rho\|_\infty$. To estimate $\|(Qu)(\alpha)\|$, it follows from the resolvent equation and (4.27) that

$$\begin{aligned} \|(Qu)(\alpha)\| &\leq \|(Q(H_A - z)^{-1}v)(\alpha)\| + \|\{Q(H_A - z)^{-1}QV(Q + P)u\}(\alpha)\| \\ &\leq C_1 B^{-1} \|v(\alpha)\| + C_1 B^{-1} M_0 \|(Qu)(\alpha)\| \\ &\quad + C_1 B^{-1} M_0 \|(QVPu)(\alpha)\|, \end{aligned} \quad (4.31)$$

with $M_0 \equiv \|V\|_\infty < \infty$. Using the estimate on QVP derived in the appendix and taking $B > 2M_0C_1$, we obtain,

$$\|(Qu)(\alpha)\| \leq 2C_1 B^{-1} \|v(\alpha)\| + C_8 B^{-3/2} \|(Pu)(\alpha)\|, \quad (4.32)$$

where $C_8 \equiv 2M_0C_1C_2$. Substituting (4.32) into (4.30), we obtain

$$(a - C_6\gamma B^{-1} - C_7C_8B^{-2})\|(Pu)(\alpha)\| \leq (C_1 + 2C_1C_7B^{-3/2})\|v(\alpha)\|. \quad (4.33)$$

This proves (4.25) for B large enough. Inserting (4.25) into (4.32) yields (4.26). \square

Corollary 4.1. *Let \mathcal{O} be an open, connected, bounded subset of \mathbb{R}^2 with smooth boundary and suppose $\mathcal{E} \subset \mathbb{R}^2 \setminus \mathcal{O}$. Let $E \in \sigma_0 \setminus \{B\}$ and assume that (B, E, V) satisfy (4.9) or (4.10) for some $a > 0$. Let $\chi_X, X = \mathcal{O}$ and \mathcal{E} , be bounded functions with support in X and s.t. $\|\chi_X\|_\infty \leq 1$. Then,*

$$\sup_{\epsilon \neq 0} \|\chi_{\mathcal{E}}(H_A + V - E - i\epsilon)^{-1}\chi_{\mathcal{O}}\| \leq C \max\{a^{-1}, B^{-1}\}e^{-\gamma d}, \quad (4.34)$$

where C and γ are as in Theorem 4.3 and $d \equiv \text{dist}(\mathcal{O}, \mathcal{E})$.

Proof. This is an immediate consequence of Theorem 4.3. We set $\rho(x) \equiv \text{dist}(x, \mathcal{O})$ and choose $v \equiv \chi_{\mathcal{O}} v$. Then, $e^{i\alpha\rho} v = v$, $\forall \alpha \in \mathbb{C}$. For u a solution of $(H_A + V - E - i\epsilon)u = \chi_{\mathcal{O}} v$, one has $\forall \alpha \in \mathbb{C}$, $|\text{Im } \alpha| < \gamma$,

$$\begin{aligned} \|\chi_{\mathcal{E}}(H_A + V - E - i\epsilon)^{-1} \chi_{\mathcal{O}} v\| &= \|\chi_{\mathcal{E}}(P + Q)u\| \\ &\leq e^{-d(\text{Im } \alpha)} \{ \|e^{\alpha\rho} P u\| + \|e^{\alpha\rho} Q u\| \} \\ &\leq e^{-d(\text{Im } \alpha)} C \max \{ a^{-1}, B^{-1} \} \|v\|, \end{aligned}$$

by Theorem 4.3. Taking $\text{Im } \alpha \rightarrow \gamma$, we obtain (4.34). \square

5 Proof of the Main Theorem

We will show below that Corollary 4.1 implies hypothesis [H1](γ_0, ℓ_0) of [8]. This hypothesis, along with Wegner's estimate, Theorem 3.1, are the main starting points of the multi-scale analysis described in [8]. The goal of this analysis is to verify the main assumption (A2) of Theorem 3.2 of [8], which gives sufficient conditions for pure point spectrum in an interval and exponential decay of eigenfunctions. A version of Kotani's trick, necessary to control the singular continuous spectrum for the models studied here, follows from Lemma 3.2.

In order to make this paper more self-contained, we recall the main points of this analysis here and refer to [8] for the details. To reduce the family H_{ω} in (2.1) to a one parameter family, we consider variations $\omega' \in \Omega$ for which only λ_0 changes. For ω fixed and $\lambda \equiv \lambda_0(\omega') - \lambda_0(\omega)$, we have

$$H_{\omega'} = H_{\omega} + \lambda u \equiv H_0 + \lambda u = H_{\lambda}, \quad (5.1)$$

with u satisfying (V1). Let $R_{\lambda}(z) \equiv (H_{\lambda} - z)^{-1}$. We first check the compactness condition, (A1), of [8]. By the diamagnetic inequality,

$$e^{-H_{\lambda}(A)t} \leq e^{-H_{\lambda}(0)t}, \quad t \geq 0 \quad (5.2)$$

and it is clear that $u^{\frac{1}{2}} e^{-H_{\lambda}(0)t} u^{\frac{1}{2}}$, $t > 0$, is compact. Using the integral representation

$$R_{\lambda}(x) = \int_0^{\infty} e^{-(H_{\lambda}(A)-x)t} dt, \quad x < 0, \quad (5.3)$$

it follows from the norm-convergence of the integral and inequality (5.2) that $u^{\frac{1}{2}}R_\lambda(z)u^{\frac{1}{2}}$ is compact for $\text{Im}z \neq 0$, $\forall \lambda$. This, for $\lambda = 0$, is condition (A1) of [8].

The second condition (A2) is that $\exists I_0 \subset I$, I some interval, and $|I_0| = |I|$ s.t. $\forall E \in I_0$,

$$\sup_{\epsilon \neq 0} \|R_0(E + i\epsilon)u^{\frac{1}{2}}\| < \infty. \quad (5.4)$$

The multi-scale analysis is used to verify this condition for a.e. ω (recall $H_0 = H_\omega$). The main theorem, which we recall in the present context, concerning condition (5.4), is the following.

Theorem 5.1 (*Theorem 2.3 of [8]*). *Let $\gamma_0 > 0$. \exists a minimum length scale $\ell^* \equiv \ell^*(\gamma_0, C_W)$, s.t. : if $[H1](\gamma_0, \ell_0)$ holds at energy E for $\ell_0 > \ell^*$, then for \mathbb{P} - a.e. $\omega \exists$ a finite constant $d_\omega > 0$ s.t.*

$$\sup_{\epsilon \neq 0} \|(H_\omega - E - i\epsilon)^{-1}u^{\frac{1}{2}}\| < d_\omega \delta(u),$$

where $\delta(u)$ depends only on u .

We prove below that $[H1](\gamma_0, \ell_0)$ holds at each energy in $[B - M_0, B - \mathcal{O}(B^{-1})] \cup I_0(B) \cap \sigma_0$ with a suitable γ_0 (see Proposition 5.1) and for all ℓ_0 large enough provided B is large. By Theorem 3.2 of [8], this theorem and the compactness result shown above imply that H_λ in (5.1) has pure point spectrum in this set for a.e. λ . By the probabilistic arguments of [8], we conclude that H_ω has only pure point spectrum in this set for \mathbb{P} - a.e. ω .

The second main theorem which we recall here allows us to prove exponential decay of the eigenfunctions.

Theorem 5.2 (*Theorem 2.4 of [8]*). *Let χ_x be the characteristic function of a unit cube centered at $x \in \mathbb{R}^2$. Under the assumptions of Theorem 5.1, for \mathbb{P} - a.e. $\omega \exists$ a finite constant $d_\omega > 0$ s.t. for all x , $\|x\|$ large enough,*

$$\sup_{\epsilon > 0} \|\chi_x(H_\omega - E - i\epsilon)^{-1}u^{\frac{1}{2}}\| \leq d_\omega e^{-\gamma_1 \|x\|},$$

where $\gamma_1 \equiv (1/6\sqrt{2})\gamma_0$, γ_0 as in Theorem 5.1.

Let us remark that for our problem, $\gamma_0 \sim B^\sigma$, for some $\sigma > 0$ so there is exponential decay in the B -field also. We now turn to the proof of [H1](γ_0, ℓ_0).

To begin, we introduce some geometry. In this section, we work with subregions of the lattice $\Gamma \equiv e^{i\pi/4}\sqrt{2}\mathbb{Z}^2$, introduced in section 4, rather than in \mathbb{Z}^2 . Recall that there is a 1:1 correspondence between bonds $b_j \in \Gamma$ and vertices $j \in \mathbb{Z}^2$. We arbitrarily choose a vertex of Γ as the origin and define boxes $\Lambda_\ell \subset \Gamma$ relative to this point,

$$\Lambda_\ell \equiv \left\{x \in \mathbb{R}^2 \mid |x_i| \leq \ell/2 \text{ for } i = 1, 2\right\}.$$

For convenience, we fix points so the bond b_0 has one of its ends at $0 \in \Gamma$. For any $\delta > 0$, consider $\Lambda_{\ell,\delta} \equiv \{x \in \Lambda_\ell \mid \text{dist}(x, \partial\Lambda_\ell) < \delta\}$. Let $\chi_{\ell,\delta}$ be the C^2 -function which satisfies $\chi_{\ell,\delta} > 0$, $|\nabla\chi_{\ell,\delta}| \subset \Lambda_\ell \setminus \Lambda_{\ell,\delta}$ and $\chi_{\ell,\delta}|_{\Lambda_{\ell,\delta}} = 1$. Let $W(\chi) \equiv [\chi, H_A]$, for any $\chi \in C^2$. Let $V_\Lambda \equiv V|_\Lambda$, $\Lambda \subset \mathbb{R}^2$ and $H_\Lambda \equiv H_A + V_\Lambda$, as in section 4.

We apply the multi-scale analysis to H_Λ relative to the lattice Γ . We verify condition [H1](γ_0, ℓ_0) of [8] using Corollary 4.1 and the geometric resolvent equation (GRE). We must show that for $E \in [B - M_0, B - \mathcal{O}(B^{-1})] \cup I_0(B) \cap \sigma_0$ and for all ℓ_0 sufficiently large, that the following holds:

[H1] (γ_0, ℓ_0) For some $\gamma_0 > 0$, $\ell_0 > 1$, $\exists \xi > 4$ s.t.

$$\mathbb{P} \left\{ \sup_{\epsilon > 0} \|W(\chi_{\ell,\delta})R_{\Lambda_{\ell_0}}(E + i\epsilon)\chi_{\ell_0/3}\| \leq e^{-\gamma_0\ell_0} \right\} \geq 1 - \ell_0^{-\xi}.$$

We begin with a simple lemma which allows us to control the gradient term in $W(\chi_{\ell,\delta})$.

Lemma 5.1 *Let $H_\Lambda \equiv (p - A)^2 + V_\Lambda$ and write $R_\Lambda \equiv R_\Lambda(E + i\epsilon)$, $\epsilon \neq 0$, $E \in \mathbb{R}$. For any $u \in L^2(\mathbb{R}^2)$, $\|u\| = 1$, we have for $i = 1, 2$,*

$$\|(p - A)_i R_\Lambda u\|^2 \leq \|R_\Lambda u\| + (2M_0 + |E|)\|R_\Lambda u\|^2, \quad (5.5)$$

where $M_0 \equiv \|V_\Lambda\|_\infty > 0$. Moreover, for any bounded $\chi \in C^1$, we have,

$$\begin{aligned} \sum_{i=1}^2 \|\chi(p - A)_i R_\Lambda u\|^2 &\leq \|\chi R_\Lambda u\| + (2M_0 + |E|)\|\chi R_\Lambda u\|^2 \\ &\quad + 2 \sum_{i=1}^2 \|(\partial_i \chi)u\| \|\chi(p - A)_i R_\Lambda u\|. \end{aligned} \quad (5.6)$$

Proof

The inequality (5.4) follows directly from the equality

$$\langle R_\Lambda u, H_A R_\Lambda u \rangle = \langle R_\Lambda u, u \rangle - \langle R_\Lambda u, (V_\Lambda - E - i\epsilon) R_\Lambda u \rangle,$$

and the Cauchy-Schwartz inequality. The inequality (5.5) follows in the same way by writing out $\|\chi(p - A)_i R_\Lambda u\|^2$. ■

We now prove the main result of this section. Recall that \mathcal{R} denotes the ribbon defined in section 4.

Proposition 5.1 *Let χ_2 be any function, $\|\chi_2\|_\infty \leq 1$, supported on $\Lambda_\ell \cap \text{Ext}\mathcal{R}$, where $\text{Ext}\mathcal{R} \equiv \{x \in \mathbb{R}^2 | \lambda x \notin \mathcal{R} \ \forall \ \lambda \geq 1\}$, so that, in particular, $\text{supp } \chi_2 \cap \mathcal{R} = \emptyset$. For any $E \in \sigma_0 \setminus \{B\}$, $\delta > 0$, $\epsilon > 0$, and $a > 0$, we have*

$$\sup_{\epsilon \neq 0} \|\chi_2 R_{\Lambda_\ell}(E + i\epsilon)\chi_{\ell/3}\| \leq C e^{-\gamma d} \max\{a^{-1}, B^{-1}\} \cdot \max\{\delta^{-1}, (2M_0 + |E|)\delta^{-2}\}, \quad (5.7)$$

where C and γ are as in Theorem 4.3 and $d \equiv r_1 - 3\epsilon$ ($r_1 \equiv \text{diam}\mathcal{R}$), with a probability larger than

$$1 - \left\{ C e^{-m\ell} + C_W [\text{dist}(E, B)]^{-2} \|g\|_\infty \delta B \ell^2 \right\}. \quad (5.8)$$

In particular, for $\chi_{\ell,\delta}$ defined above and $E \in \sigma_0$ with $a = \frac{E-B}{2} = \mathcal{O}(B^{-1+\sigma})$, any $\sigma > 0$, we have that for any $\ell_0 > \sqrt{2}$ and large enough, and any $\xi > 4$, $\exists B(\ell_0) > 0$ s.t. $\forall B > B(\ell_0)$, [H1] (γ_0, ℓ_0) holds for some $\gamma_0 > \gamma d/4\ell_0 > 0$, so that $\gamma_0 = \mathcal{O}\{\min(B^{1/2}, B^\sigma)\}$.

Proof

1. By Corollary 4.1, $\exists B_0$ s.t. $B > B_0$ implies \exists a ribbon $\mathcal{R} \subset \Lambda_\ell \setminus \Lambda_{\ell/3}$ (with a probability given by (4.7)) satisfying

$$\text{dist}(\mathcal{R}, \partial\Lambda_\ell), \text{ and } \text{dist}(\mathcal{R}, \partial\Lambda_{\ell/3}) > \frac{1}{\sqrt{2}} + r_u > 0, \quad (5.9)$$

and

$$r_1 \equiv \text{diam}\mathcal{R} > 2 \left(\frac{1}{\sqrt{2}} - r_u \right), \quad (5.10)$$

and such that

$$V(x) + B - E > -a \quad \forall x \in \mathcal{R}, \quad a = \frac{E - B}{2}. \quad (5.11)$$

(We assume $E > B$; similar arguments hold for $E < B$). For any $\epsilon > 0$, $3\epsilon \ll r_1$, define the border of \mathcal{R} by

$$\mathcal{R}_\epsilon \equiv \{x \in \mathcal{R} \mid \text{dist}(x, \partial\mathcal{R}) < \epsilon\}.$$

Then $\mathcal{R}_\epsilon \equiv \mathcal{R}_\epsilon^+ \cup \mathcal{R}_\epsilon^-$, where \mathcal{R}_ϵ^\pm are two disjoint, connected subsets of \mathcal{R} . Let $\mathcal{C} \equiv \{x \in \mathcal{R} \mid \text{dist}(x, \mathcal{R}_\epsilon^+) = \text{dist}(x, \mathcal{R}_\epsilon^-)\}$; \mathcal{C} is a closed, connected path in \mathcal{R} . Let $\mathcal{C}_\epsilon \equiv \{x \in \mathcal{R} \mid \text{dist}(x, \mathcal{C}) < \epsilon\} \subset \mathcal{R}$ and

$$\text{dist}(\mathcal{C}_\epsilon, \mathcal{R}_\epsilon^\pm) \geq r_1 - 3\epsilon. \quad (5.12)$$

This is strictly positive. Because of this, we can adjust \mathcal{C}_ϵ so that $\partial\mathcal{C}_\epsilon$ is smooth. We need two, C^2 , positive cut-off functions. Let $\chi_{\mathcal{R}} > 0$ satisfy $\chi_{\mathcal{R}}|_{\mathcal{C}_\epsilon} = 1$ and $\text{supp} |\nabla \chi_{\mathcal{R}}| \subset \mathcal{R}_\epsilon$. Let χ_1 satisfy $\chi_1|_{\Lambda_{\ell/3}} = 1$ and $\text{supp} |\nabla \chi_1| \subset \mathcal{C}_\epsilon$ (see Figure 4). By simple commutation, we have (with χ_2 as in the proposition),

$$\begin{aligned} \chi_2 R_{\Lambda_\ell} (E + i\epsilon) \chi_{\ell/3} &= \chi_2 R_{\Lambda_\ell} \chi_1 \chi_{\ell/3} \\ &= \chi_2 R_{\Lambda_\ell} W(\chi_1) R_{\Lambda_\ell} \chi_{\ell/3} \\ &= \chi_2 R_{\Lambda_\ell} \chi_{\mathcal{R}} W(\chi_1) R_{\Lambda_\ell} \chi_{\ell/3}. \end{aligned} \quad (5.13)$$

Next, denote by $R_{\mathcal{R}}$ the resolvent of $H_{\mathcal{R}}$ defined in section 4.2. The *GRE* relating R_{Λ_ℓ} and $R_{\mathcal{R}}$ is

$$R_{\Lambda_\ell} \chi_{\mathcal{R}} = \chi_{\mathcal{R}} R_{\mathcal{R}} + R_{\Lambda_\ell} W(\chi_{\mathcal{R}}) R_{\mathcal{R}}. \quad (5.14)$$

Substituting (5.13) into (5.12) and noting that $\chi_2 \chi_{\mathcal{R}} = 0$, we obtain

$$\chi_2 R_{\Lambda_\ell} \chi_{\ell/3} = \chi_2 R_{\Lambda_\ell} W(\chi_{\mathcal{R}}) R_{\mathcal{R}} W(\chi_1) R_{\Lambda_\ell} \chi_{\ell/3}. \quad (5.15)$$

Note that from (5.11) and the choice of $\chi_{\mathcal{R}}$ and χ_1 , we obtain that

$$\text{dist}(\text{supp } W(\chi_{\mathcal{R}}), \text{supp } W(\chi_1)) \geq r_1 - 3\epsilon. \quad (5.16)$$

We apply Wegner's estimate, Theorem 3.1, to control the two R_{Λ_ℓ} factors in (5.14), and the decay estimate, Corollary 4.1, to control the factor $R_{\mathcal{R}}$, which is possible due to the localization of $W(\chi_{\mathcal{R}})$ and $W(\chi_1)$ and (5.16).

2. To estimate the $R_{\mathcal{R}}(E + i\epsilon)$ contribution, we use Corollary 4.1 with $\mathcal{O} \equiv \mathcal{C}_\epsilon$ and $\mathcal{E} = \mathcal{R}_\epsilon$. Let χ_X , $X = \mathcal{O}$ and \mathcal{E} , be a characteristic function on these sets. Then $W(\chi_{\mathcal{R}})\chi_{\mathcal{E}} = W(\chi_{\mathcal{R}})$ and $\chi_{\mathcal{O}}W(\chi_1) = W(\chi_1)$. Inserting these localization functions into (5.15), we obtain from Corollary 4.1,

$$\|\chi_{\mathcal{E}}R_{\mathcal{R}}(E + i\epsilon)\chi_{\mathcal{O}}\| \leq C \max\{a^{-1}, B^{-1}\} e^{-\gamma d}, \quad (5.17)$$

with probability larger than

$$1 - Ce^{-m\ell}, \quad (5.18)$$

for some $m = m(1 - p) > 0$ and $0 < C < \infty$. The factor d satisfies

$$d \geq r_1 - 3\epsilon, \quad (5.19)$$

where $r_1 \equiv \text{diam}\mathcal{R}$ as in (5.10).

3. Next, we turn to

$$W(\chi_1)R_\ell(E + i\epsilon)\chi_{\ell/3}, \quad (5.20)$$

and

$$\chi_2 R_\ell(E + i\epsilon)W(\chi_{\mathcal{R}}), \quad (5.21)$$

where we write R_ℓ for R_{Λ_ℓ} for short. We will bound R_ℓ by Wegner's estimate and the terms (5.20)-(5.21) via Lemma 5.1. From Theorem 3.1, we have for any $\delta > 0$,

$$\|R_\ell(E + i\epsilon)\| < \delta^{-1}, \quad (5.22)$$

with probability larger than

$$1 - C_W[\text{dist}(E, B)]^{-2} \|g\|_\infty \delta B \ell^2. \quad (5.23)$$

From (5.22) and Lemma 5.1, both (5.20) and (5.21) are bounded above by

$$2^{\frac{1}{2}} \max\left\{\delta^{-\frac{1}{2}}, (2M + |E|)^{\frac{1}{2}} \delta^{-1}\right\}, \quad (5.24)$$

with probability at least (5.23).

4. Using the estimate $P(A \cap B) \geq P(A) + P(B) - 1$, and (5.17)-(5.18) and (5.23)-(5.24), we find

$$\begin{aligned} \|\chi_2 R_\ell(E + i\epsilon)\chi_{\ell/3}\| &\leq 2C \max\{a^{-1}, B^{-1}\} \cdot \max\{\delta^{-1}, \\ &\quad (2M_0 + |E|)\delta^{-2}\} \cdot e^{-\gamma d}, \end{aligned} \quad (5.25)$$

with probability at least

$$1 - \left\{ C e^{-m\ell} + C_W [\text{dist}(E, B)]^{-2} \|g\|_\infty \delta B \ell^2 \right\}. \quad (5.26)$$

This proves the first part of the proposition.

5. To estimate $W(\chi_{\ell,\delta}) R_\ell \chi_{\ell/3}$, we use the second formula of Lemma 5.1, (5.6), which gives

$$\begin{aligned} \|\chi_2(p - A)_i R_\ell \chi_{\ell/3}\|^2 &\leq \|\chi_2 R_\ell \chi_{\ell/3}\|^2 + (2M + |E|) \|\chi_2 R_\ell \chi_{\ell/3}\|^2 \\ &\quad + 2 \max_{i=1,2} \|(\partial_i \chi_2) R_\ell \chi_{\ell/3}\| \|\chi_2(p - A)_i R_\ell \chi_{\ell/3}\|. \end{aligned} \quad (5.27)$$

Since $\partial_i \chi_2$ satisfies the same condition as χ_2 , the factor $\|(\partial_i \chi_2) R_\ell \chi_{\ell/3}\|$ in (5.27) satisfies the estimate (5.24) with possibly a different constant. Solving the quadratic inequality (5.27), we obtain

$$\begin{aligned} \|\chi_2(p - A)_i R_\ell \chi_{\ell/3}\| &\leq \max_{i=1,2} \left\{ \|(\partial_i \chi_2) R_\ell \chi_{\ell/3}\| \right. \\ &\quad + \left[\|(\partial_i \chi_2) R_\ell \chi_{\ell/3}\|^2 + \left(\|\chi_2 R_\ell \chi_{\ell/3}\| \right. \right. \\ &\quad \left. \left. + (2M_0 + |E|) \|\chi_2 R_\ell \chi_{\ell/3}\|^2 \right)^{1/2} \right] \left. \right\}, \end{aligned} \quad (5.28)$$

which can be estimated as in (5.25). Finally, we write

$$\|W(\chi_{\ell,\delta}) R_\ell \chi_{\ell/3}\| \leq \|(\Delta \chi_{\ell,\delta}) R_\ell \chi_{\ell/3}\| + 2 \sum_{j=1}^2 \|(\partial_j \chi_{\ell,\delta})(p - A)_j R_\ell \chi_{\ell/3}\|, \quad (5.29)$$

which can be estimated from (5.25) with $\chi_2 \equiv \Delta \chi_{\ell,\delta}$ and (5.27) with $\chi_2 \equiv (\partial_j \chi_{\ell,\delta})$.

6. We now show that for any ℓ_0 large enough, $\exists B_0 \equiv B_0(\ell_0)$ such that for all $B > B_0$, condition $[H1](\ell_0, \gamma_0)$ is satisfied with $\gamma_0 = \mathcal{O}\{\min(B^{1/2}, B^\sigma)\}$. We take $E \in [B - M_0, B - \mathcal{O}(B^{-1})] \cup I_0(B) \cap \sigma_0$ and $a = \frac{E-B}{2} = \mathcal{O}(B^{-1+\sigma})$, for any $\sigma > 0$. First, we require that (5.25) be bounded above by $e^{-\gamma d/2}$. This leads to the condition

$$C\delta^{-2}B^{2-\sigma}e^{-\gamma d} \leq e^{-\gamma d/2}, \quad (5.30)$$

where $\gamma = C_2 \min\{B^{\frac{1}{2}}, B^\sigma\}$. This condition implies that we must choose δ in (5.22) to satisfy

$$\delta > B^{1-(\sigma/2)}e^{-\gamma d/4}. \quad (5.31)$$

If we now define

$$\gamma_0 \equiv \gamma d/4\ell_0$$

we find that

$$||W(\chi_{\ell,\delta}) R_{\ell\chi_{\ell/3}}|| \leq e^{-\gamma_0 \ell_0}.$$

Next, the probability estimate (5.26) leads to the condition

$$Ce^{-m\ell_0} + C_2B^{3-2\sigma}\delta\ell_0^2 \leq \ell_0^{-\xi}, \quad (5.32)$$

or, for all ℓ_0 large,

$$C_3B^{3-2\sigma}\delta\ell_0^2 \leq \ell_0^{-\xi}, \quad (5.33)$$

for some $\xi > 4$. We can choose δ so that both conditions (5.31) and (5.33) are satisfied provided the condition

$$\ell_0^{\xi+2} < B^{3/2-(5/2)\sigma}e^{\gamma d/4}, \quad (5.34)$$

is satisfied for some $\xi > 4$. It is clear from the definition of γ , that for any ℓ_0 , there exists a $B_0 \equiv B_0(\ell_0)$ such that condition (5.34) is satisfied for all $B > B_0$. This completes the proof of the theorem. ■

6 Appendix

The following estimates hold for all B sufficiently large.

Lemma A.1. *Let $V \in C_b^2(\mathbb{R})$. \exists constant $C > 0$ depending only on $\|\partial^\alpha V\|_\infty$, $|\alpha| = 0, 1, 2$, such that $\forall \alpha \in S$,*

$$\|P(\alpha)VQ(\alpha)\| \leq CB^{-1/2}. \quad (6.1)$$

Proof. Let $z \equiv B - 1$, so $z \in \rho(H_A(\alpha))$ for $\alpha \in S$. We have

$$\begin{aligned} P(\alpha)VQ(\alpha) &= P(\alpha)(H_A(\alpha) - z)(H_A(\alpha) - z)^{-1}VQ(\alpha) \\ &= P(\alpha)(H_A(\alpha) - z)V(H_A(\alpha) - z)^{-1}Q(\alpha) \\ &\quad + P(\alpha)(H_A(\alpha) - z)(H_A(\alpha) - z)^{-1}[V, H_A(\alpha)](H_A(\alpha) - z)^{-1}Q(\alpha). \end{aligned} \quad (6.2)$$

Recall that $P(\alpha)$ is analytic in $\alpha \in S$. As

$$\begin{aligned} P(\alpha)(H_A(\alpha) - z) &= P(\alpha)(B - z) \\ &= P(\alpha), \end{aligned} \quad (6.3)$$

for $\alpha \in \mathbb{R}$, the identity principle for analytic functions implies this holds for $\alpha \in S$. This result (A.3) and estimates (4.16)-(4.17) imply that the first term on the right in (A.2) is bounded as

$$\|P(\alpha)(H_A(\alpha) - z)V(H_A(\alpha) - z)^{-1}Q(\alpha)\| \leq C_0\|V\|_\infty B^{-1},$$

$\forall \alpha \in S$. As for the second term, the commutator is (see (4.18))

$$[V, H_A(\alpha)] = 2i(p - \alpha \nabla \rho - A) \cdot \nabla V - \Delta V.$$

The resulting term in (A.2) involving ΔV is treated as above. As for the derivative term, it suffices to show

$$\|(H_A(\alpha) - z)^{-1}(p_i - \alpha \partial_i \rho - A_i)\| \leq C_1 B^{1/2}, \quad (6.4)$$

for all $\alpha \in S$. To see this, let $V_i(\alpha) \equiv (p_i - \alpha \partial_i \rho - A_i)$ and $R(\alpha) \equiv (H_A(\alpha) - z)^{-1}$. For $u = R(\alpha)v$, $v \in L^2(\mathbb{R}^2)$, we have

$$\begin{aligned} \sum_{i=1}^2 \|V_i(\alpha)u\|^2 &= \langle u, \{H_A(\alpha) + 2i(\operatorname{Im} \alpha) \nabla \rho \cdot V(\alpha)\}u \rangle \\ &= \langle R(\alpha)v, v \rangle + z\|u\|^2 + 2i(\operatorname{Im} \alpha) \langle u, \nabla \rho \cdot V(\alpha)u \rangle. \end{aligned}$$

This leads to a quadratic inequality for each $i = 1, 2$,

$$\begin{aligned} \|V_i(\alpha)R(\alpha)v\|^2 &\leq \|R(\alpha)v\|(\|v\| + |z| \|R(\alpha)v\|) \\ &\quad + 2|\operatorname{Im} \alpha| \|\nabla \rho\|_\infty \|R(\alpha)v\| \left\{ \max_{j=1,2} \|V_j(\alpha) \right. \\ &\quad \left. \times R(\alpha)v\| \right\}. \end{aligned}$$

Solving this, and noting that $|\operatorname{Im} \alpha| \leq B^{1/2}$, $|z| = \mathcal{O}(B)$, and, for this z , $\|R(\alpha)\| < C_0$ by (4.16)-(4.17), we get

$$\|V_i(\alpha)R(\alpha)v\| \leq C_1 B^{1/2}, \quad (6.5)$$

which is (A.4). \square

Lemma A.2. *Let ρ be the distance function defined in section 4. \exists constant $C > 0$ depending only on $\|\partial^\alpha \rho\|_\infty$, $|\alpha| = 0, 1, 2$, such that $\forall \alpha \in S$,*

$$\|P(\alpha)^*Q(\alpha)\| \leq CB^{-1/2}|\operatorname{Im} \alpha|, \quad (6.6)$$

for $|\operatorname{Im} \alpha| \leq B^{1/2}$.

Proof. We can assume that α is purely imaginary by a standard unitary equivalence argument (note that when α is real, $\|P(\alpha)^*Q(\alpha)\| = 0$.) Let $z \equiv B - 1$, as in Lemma A.1. We then have by (6.3),

$$\begin{aligned} P(\alpha)^*Q(\alpha) &= P(\alpha)^*(H_A^*(\alpha) - z)^{-1}Q(\alpha) \\ &= P(\alpha)^*\{(H_A^*(\alpha) - z)^{-1} - (H_A(\alpha) - z)^{-1}\}Q(\alpha) \\ &\quad + P(\alpha)^*(H_A(\alpha) - z)^{-1}Q(\alpha). \end{aligned} \quad (6.7)$$

The last term is $\mathcal{O}(B^{-1})$ by (4.16)–(4.17). Let us write $\alpha \equiv i\beta$, with β real. Then by the resolvent equation, we have

$$(H_A^*(\alpha) - z)^{-1} - (H_A(\alpha) - z)^{-1} = -2i\beta(H_A^*(\alpha) - z)^{-1}[2(p - A) \cdot \nabla \rho + i\Delta \rho](H_A(\alpha) - z)^{-1}. \quad (6.8)$$

Using (6.8) in (6.7), we obtain the bound,

$$\begin{aligned} \|P(\alpha)^* Q(\alpha)\| &\leq 2|Im(\alpha)| \|\{2(p - A) \cdot \nabla \rho + i\Delta \rho\} P^*(\alpha)\| \|(H_A(\alpha) - z)^{-1}\} Q(\alpha) \\ &\leq C_1 |Im(\alpha)| B^{-1} (\|2(p - A) \cdot \nabla \rho P^*(\alpha)\| + C_2), \end{aligned} \quad (6.9)$$

where the constants are bounded as in the lemma. We again used (4.16)–(4.17) and the boundedness of $\Delta \rho$. The proof will follow from (6.8) once we show that

$$\| |p - A| P^*(\alpha) \| \leq C_1 B^{1/2}. \quad (6.10)$$

Inequality (6.10) follows directly as in (6.4) if we write

$$(H_A^*(\alpha) - z)^{-1} (p_i - A_i) = (H_A^*(\alpha) - z)^{-1} \{ (p_i + i\beta \partial_i \rho - A_i) - i\beta \partial_i \rho \}, \quad (6.11)$$

and note that $\beta \leq B^{1/2}$. This proves (6.10). \square

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